Lemmas on Determination of Minimum Bandwidth of Q-filter for Robust Stability of Feedback Loop with Disturbance Observers

Hamin Chang, Hyuntae Kim, and Hyungbo Shim

Abstract— **As the disturbance observer (DOB)-based controller has been widely applied in practice, various aspects of the disturbance observer have been theoretically studied. In particular, robust stability of the closed-loop system with Q-filter-based DOB has been rigorously analyzed, and finally, a necessary and sufficient condition for robust stability was obtained under the premise that the bandwidth of Q-filter is large. However, even the most recent study about the design of Q-filter-based DOB for robust stability does not offer a practical method for the determination of the Q-filter's bandwidth. In this paper, we present several lemmas regarding the determination of the bandwidth, with which the linear closed-loop system remains stable against arbitrarily large but bounded parametric uncertainties. In particular, our study proposes a procedure to determine the threshold such that robust stability is lost if the bandwidth of the Q-filter becomes lower than the threshold. The proposed procedure can also be used even for designing the Q-filter-based DOB for non-minimum phase systems.**

I. Introduction

The Q-filter-based disturbance observer (DOB) has been a powerful robust control scheme to reject disturbances and compensate plant uncertainties since it was first introduced by [1]. The DOB has been frequently employed in the industry from the time when it was regarded as a rather heuristic method, and now several theories are available about the robust stability of the DOB-based control systems. Among others, [2] and [3] introduced singular perturbation theory into the analysis of DOB-based control systems, and this insight yielded a necessary and sufficient condition for robust stability [4]. This finding, in turn, enabled the systematic design of DOBs for robust stability against arbitrarily large parameter variations.

Based on the robust stability result, more insights about the DOB have been discovered. For instance, it was found that a high-gain observer is already embedded in the seemingly different structure of the Q-filter-based DOB and that the zero-dynamics of the plant becomes decoupled when the DOB is installed in the feedback loop [5]. This finding provides an insightful explanation for how the DOB works as a robust controller, by which both the benefits and the limitations of DOB are clarified. It was also figured out how imprecise identification of the relative degree of an uncertain plant affects stability [6] and how the classical measure of robustness, the gain/phase margin, is affected by a DOB in the loop [7].

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All authors are with ASRI, School of Electrical Engineering and Computer Science, Seoul National Univ., Kwanak P.O.Box 34, Seoul, 151-600, Korea. Corresponding author: hshim@snu.ac.kr

Based on these analyses, a few modified DOBs are also proposed to overcome the limitations of the classical DOB. For example, a way to modify the classical DOB for robust transient performance was presented in [3], and a way to embed an internal model that generates external disturbances so that the modeled disturbances are rejected perfectly while the unmodeled disturbances are attenuated at the desired level was presented in [8]. On top of those theoretical developments, the DOB is replacing traditional robust control methods. Examples include flight control of drones [9], platooning of multi-vehicles [10], load-frequency control of power-grid [11], robustifying the reinforcement learning based controller [12], and even generating stealthy attacking signals for control systems [13].

However, most of these results are based on the premise that the bandwidth of the Q-filter is sufficiently large. For example, the necessary and sufficient condition for robust stability in [4] is derived when the time constant τ of the Q-filter is less than a threshold τ^*. While the threshold τ^* is presented in [4], it is just a conservative value, and in practice, the selection of τ^* should be obtained by a repeated simulation or by trial and error.

In this paper, we study how to choose the minimum bandwidth of Q-filter, i.e., the non-conservative value of τ^*, under which robust stability is guaranteed against parameter uncertainties within prescribed ranges. Having non-conservative τ^* is desirable because there might exist unavoidable physical constraints that limit the available bandwidth of the Q-filter. The existence of unmodeled dynamics in the model of the plant is another reason why we need to avoid unnecessarily large bandwidth of Q-filter. Moreover, succinct computation of τ^* is desired, which does not rely on an iterative method. In this paper, a few lemmas are presented with which exact computation of τ^* is enabled. This work will pave a road to building a computer-aided toolbox for designing DOBs that are robust against given uncertain variation of parameters.

Finally, it will be shown that the proposed procedure to find the minimum bandwidth, or the value of τ^*, of the Q-filter, can also be used for finding suitable bandwidths or the values τ even for non-minimum phase plants. No universal design methods of DOB for non-minimum phase plants are available yet. However, since we are using a numerical method whatsoever, DOB can be designed regardless of whether the plant is of minimum phase or not.

This paper is organized as follows. An overview of the Q-filter-based DOB is briefly explained in Section II. Section III describes a couple of assumptions and primary results of [4] as preliminary. In Section IV, we propose several

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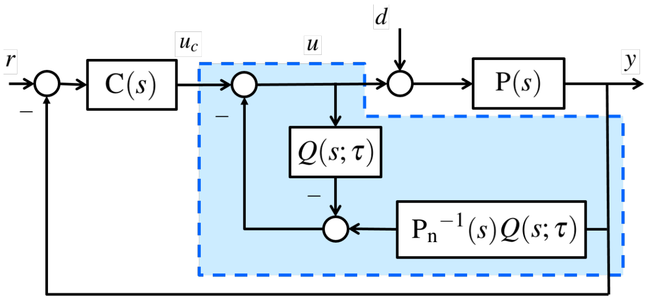

Fig. 1. Block diagram of the closed-loop system with Q-filter-based DOB (sky-blue dashed block).

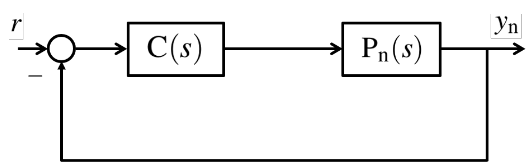

Fig. 2. Block diagram of the nominal closed-loop system.

necessary and sufficient conditions for the robust stability of the DOB-based control system as lemmas and then show how we find an appropriate bandwidth of the Q-filter based on the suggested lemmas. Some illustrative examples that demonstrate the usefulness of the lemmas are given in Section V. Finally, this paper is summarized and concluded in Section VI.

II. Overview of Q-filter-based Disturbance Observer

The standard structure of the Q-filter-based DOB and the closed-loop system are depicted in Fig. 1. In the figure, $\mathrm{P}(s)$ and $\mathrm{P_n}(s)$ represent the real plant and its nominal model, respectively, $\mathrm{C}(s)$ is a proper (implementable) controller which is usually designed a priori for $\mathrm{P_n}(s)$, and $Q(s;\tau)$ is a stable low-pass filter called Q-filter with a parameter τ. This paper focuses on the design of the suitable value τ that decides the time constant or the bandwidth of the Q-filter for robust stability of the closed-loop system against a given variation of uncertain parameters. It is well-known that, if the reference r and disturbance d consist of low-frequency components and if all other parameters of Q-filter are properly set, then the Q-filter-based DOB with a large bandwidth of the Q-filter (that is, a small magnitude of τ) enables the system in Fig. 1 to approximate the nominal closed-loop system in Fig. 2 (see, e.g., [5]). In other words, the following approximation

$$y(j\omega) \approx \frac{\mathrm{P_n}(j\omega)\mathrm{C}(j\omega)}{1+\mathrm{P_n}(j\omega)\mathrm{C}(j\omega)} r(j\omega) = y_{\mathrm{n}}(j\omega)$$

holds with a sufficiently large bandwidth of the Q-filter, where y and y_{n} are the outputs of the DOB-based control system in Fig. 1 and the nominal closed-loop system in Fig. 2, respectively. This capability of approximation is one of the main features of the Q-filter-based DOB scheme, which is often called *nominal performance recovery*.

III. Preliminary

In this paper, parametric uncertainty of the plant $\mathrm{P}(s)$ is assumed to satisfy the following.

Assumption 1: The real plant $\mathrm{P}(s)$ and its nominal model $\mathrm{P_n}(s)$ belong to the set of uncertain plants:

$$\begin{aligned}\mathscr{P} := \Bigg\{ \mathrm{P}(s) = \frac{\beta_{n-\nu}s^{n-\nu} + \beta_{n-\nu-1}s^{n-\nu-1} + \ldots + \beta_0}{\alpha_n s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_0} \\ : \alpha_i \in [\underline{\alpha}_i, \overline{\alpha}_i], \beta_i \in [\underline{\beta}_i, \overline{\beta}_i] \Bigg\},\end{aligned} \tag{1}$$

where n and ν are positive integers such that $n \geq \nu$ and $\underline{\alpha}_i$, $\overline{\alpha}_i$, $\underline{\beta}_i$, and $\overline{\beta}_i$ are known constants such that $[\underline{\alpha}_n, \overline{\alpha}_n]$, $[\underline{\beta}_{n-\nu}, \overline{\beta}_{n-\nu}] \subset (0,\infty)$, where $(0,\infty)$ denotes the positive real line. □

In the assumption, the condition $[\underline{\alpha}_n, \overline{\alpha}_n]$, $[\underline{\beta}_{n-\nu}, \overline{\beta}_{n-\nu}] \subset (0,\infty)$ implies that all the plants in the set have the same relative degree and have the same sign of the high frequency gain (which is positive, without loss of generality). From the assumption, it is clear that the set $\mathscr{P}$ incorporates arbitrarily large but bounded uncertainties of the parameters. Note that the description of the set $\mathscr{P}$ in (1) has redundancy. This redundancy disappears by letting, for example, $\underline{\alpha}_n = \overline{\alpha}_n = 1$, but for the general purpose, we let all the parameters are independent of one another.

The stable low-pass filter $Q(s;\tau)$ is usually designed in the form

$$\begin{aligned}Q(s;\tau) &= \frac{c_k(\tau s)^k + c_{k-1}(\tau s)^{k-1} + \cdots + c_0}{(\tau s)^l + a_{l-1}(\tau s)^{l-1} + \cdots + a_1(\tau s) + a_0} \\ &=: \frac{N_Q(s;\tau)}{D_Q(s;\tau)},\end{aligned} \tag{2}$$

where N_Q and D_Q indicates polynomials that are the numerator and the denominator of $Q(s;\tau)$, respectively. The real positive τ determines the time constant or the bandwidth, $a_0 = c_0$ for the unity dc gain, and k and l are some non-negative integers such that $k \leq l-\nu$, where ν is the relative degree of $\mathrm{P_n}(s)$. Furthermore, we assume the following necessary condition (see [4]), which is relevant to $\mathrm{P_n}(s)$, $\mathrm{C}(s)$, and $Q(s;\tau)$, for robust stability under large bandwidth of Q-filter.

Assumption 2: The nominal closed-loop system $\mathrm{P_n C}/(1+\mathrm{P_n C})$ is internally stable, and the polynomial

$$p_f(s) := D_Q(s;1) + \left(\lim_{s\to\infty} \frac{\mathrm{P}(s)}{\mathrm{P_n}(s)} - 1 \right) N_Q(s;1)$$

is Hurwitz for all $\mathrm{P}(s) \in \mathscr{P}$ in (1). □

Note that $\tau = 1$ in the assumption, and thus, the assumption is independent of the choice of τ. In fact, a systematic way to choose the parameters a_i and c_i of the Q-filter in (2) such that the second condition of Assumption 2 holds has been presented in [4, Sec. 2.3].

The following theorem, taken from [4], plays a crucial role to design Q-filter-based DOB for robust stability of the closed-loop system in Fig. 1.

Theorem 1: Suppose that Assumptions 1 and 2 holds. If all the plants $\mathrm{P}(s) \in \mathscr{P}$ are of minimum phase, then there

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exists a constant τ^* such that, for all $0<\tau<\tau^*$, the closed-loop system in Fig. 1 is robustly internally stable (against the uncertainty of $\mathscr{P}$).

On the contrary, if $\mathscr{P}$ contains a non-minimum phase plant such that at least one zero has positive real parts, then there is τ^* such that, for all $0<\tau<\tau^*$, the closed-loop system is not robustly internally stable. □

While the former part of Theorem 1 guarantees the existence of the threshold τ^* (or, the minimum bandwidth of Q-filter), its proof in [4] presents a conservative choice of τ^*. In fact, no method to find the exact and non-conservative value of τ^* has been reported in the literature yet. In the next section, some useful lemmas are introduced which can be utilized to obtain the exact value of τ^* under Assumptions 1 and 2.

IV. Main Result

In this section, we first discuss the characteristic polynomial of the closed-loop system in Fig. 1, and then we provide a couple of equivalent statements which are necessary and sufficient conditions for robust stability of the DOB-based control system. In addition, we present a procedure to compute the value of τ^* in Theorem 1 based on the given lemmas and discuss the case when non-minimum phase systems belong to the plant set $\mathscr{P}$.

Let $\mathrm{P}(s)$, $\mathrm{P_n}(s)$, and $\mathrm{C}(s)$ be represented by the ratios of coprime polynomials such as $\mathrm{P}(s)=N(s)/D(s)$, $\mathrm{P_n}(s)=N_\mathrm{n}(s)/D_\mathrm{n}(s)$, and $\mathrm{C}(s)=N_\mathrm{C}(s)/D_\mathrm{C}(s)$. Then, it is well-known that the following lemma holds [4].

Lemma 2: The closed-loop system in Fig. 1 is robustly internally stable if and only if the characteristic polynomial

$$\begin{aligned}\delta(s;\tau)=&\Big(D(s)D_\mathrm{C}(s)+N(s)N_\mathrm{C}(s)\Big)N_\mathrm{n}(s)D_Q(s;\tau)\\&+N_Q(s;\tau)D_\mathrm{C}(s)\Big(N(s)D_\mathrm{n}(s)-N_\mathrm{n}(s)D(s)\Big)\end{aligned}\tag{3}$$

is Hurwitz for all $\mathrm{P}(s)\in\mathscr{P}$ in (1). □

The characteristic polynomial $\delta(s;\tau)$ is the main concern throughout this paper. In the following subsection, a couple of equivalent statements of Lemma 2 are presented.

A. Application of Edge Theorem and Bialas' Theorem

Now, we are going to define 'polytope of polynomials', 'edge polynomial', and 'exposed edge polynomial' with respect to $\delta(s;\tau)$ in (3) to make use of the Edge theorem in [14]. For a precise definition of such terminologies above, it is recommended to refer to [14].

With uncertain polynomials $D(s)$ and $N(s)$, the characteristic polynomial $\delta(s;\tau)$ in (3) can be rewritten as

$$\begin{aligned}\delta(s;\tau)&=p_D(s;\tau)\cdot D(s)+p_N(s;\tau)\cdot N(s),\\&=p_D(s;\tau)\cdot\sum_{i=0}^{n}\alpha_i s^i+p_N(s;\tau)\cdot\sum_{j=0}^{n-\nu}\beta_j s^j,\end{aligned}$$

where $p_D(s;\tau)=D_\mathrm{C}(s)N_\mathrm{n}(s)D_Q(s;\tau)-N_Q(s;\tau)D_\mathrm{C}(s)N_\mathrm{n}(s)$ and $p_N(s;\tau)=N_\mathrm{C}(s)N_\mathrm{n}(s)D_Q(s;\tau)+N_Q(s;\tau)D_\mathrm{C}(s)D_\mathrm{n}(s)$ which are not uncertain. At this point, we define

$$\Omega:=\{\delta(s;\tau):\mathrm{P}(s)\in\mathscr{P}\}\tag{4}$$

as the set of all characteristic polynomials corresponding to every possible plants $\mathrm{P}(s)\in\mathscr{P}$ in (1). Then for each $\alpha_i^{(1)}$, $\alpha_i^{(2)}\in[\underline{\alpha}_i,\overline{\alpha}_i]$ and $\beta_j^{(1)}$, $\beta_j^{(2)}\in[\underline{\beta}_j,\overline{\beta}_j]$, where $i=0,1\cdots n$ and $j=0,1,\cdots n-\nu$, the polynomials

$$\omega_1:=p_D(s;\tau)\cdot\sum_{i=0}^{n}\alpha_i^{(1)}s^i+p_N(s;\tau)\cdot\sum_{j=0}^{n-\nu}\beta_j^{(1)}s^j,$$

$$\omega_2:=p_D(s;\tau)\cdot\sum_{i=0}^{n}\alpha_i^{(2)}s^i+p_N(s;\tau)\cdot\sum_{j=0}^{n-\nu}\beta_j^{(2)}s^j,$$

belong to Ω. Moreover, for any $\lambda\in[0,1]$, the convex combination

$$\begin{aligned}\lambda\omega_1+(1-\lambda)\omega_2=&\,p_D(s;\tau)\cdot\sum_{i=0}^{n}\Big(\lambda\alpha_i^{(1)}+(1-\lambda)\alpha_i^{(2)}\Big)s^i\\&+p_N(s;\tau)\cdot\sum_{j=0}^{n-\nu}\Big(\lambda\beta_j^{(1)}+(1-\lambda)\beta_j^{(2)}\Big)s^j,\end{aligned}$$

is also in Ω because $\lambda\alpha_i^{(1)}+(1-\lambda)\alpha_i^{(2)}\in[\underline{\alpha}_i,\overline{\alpha}_i]$ and $\lambda\beta_j^{(1)}+(1-\lambda)\beta_j^{(2)}\in[\underline{\beta}_j,\overline{\beta}_j]$. Thus, the set Ω in (4) is a polytope of polynomials, namely the convex hull of a finite number of polynomials. Indeed, we have

$$m:=2^{2n-\nu+2}$$

polynomials depending on $\underline{\alpha}_i$, $\overline{\alpha}_i$, $\underline{\beta}_j$, and $\overline{\beta}_j$, where $i=0,1\cdots n$ and $j=0,1,\cdots n-\nu$, as

$$\begin{aligned}\delta_1(s;\tau):=&\,(\underline{\alpha}_n s^n+\underline{\alpha}_{n-1}s^{n-1}+\cdots+\underline{\alpha}_0)\cdot p_D(s;\tau)\\&+(\underline{\beta}_{n-\nu}s^{n-\nu}+\underline{\beta}_{n-\nu-1}s^{n-\nu-1}+\cdots+\underline{\beta}_0)\cdot p_N(s;\tau),\\\delta_2(s;\tau):=&\,(\overline{\alpha}_n s^n+\underline{\alpha}_{n-1}s^{n-1}+\cdots+\underline{\alpha}_0)\cdot p_D(s;\tau)\\&+(\underline{\beta}_{n-\nu}s^{n-\nu}+\underline{\beta}_{n-\nu-1}s^{n-\nu-1}+\cdots+\underline{\beta}_0)\cdot p_N(s;\tau),\\\delta_3(s;\tau):=&\,(\overline{\alpha}_n s^n+\overline{\alpha}_{n-1}s^{n-1}+\cdots+\underline{\alpha}_0)\cdot p_D(s;\tau)\\&+(\underline{\beta}_{n-\nu}s^{n-\nu}+\underline{\beta}_{n-\nu-1}s^{n-\nu-1}+\cdots+\underline{\beta}_0)\cdot p_N(s;\tau),\\&\vdots\\\delta_m(s;\tau):=&\,(\overline{\alpha}_n s^n+\overline{\alpha}_{n-1}s^{n-1}+\cdots+\overline{\alpha}_0)\cdot p_D(s;\tau)\\&+(\overline{\beta}_{n-\nu}s^{n-\nu}+\overline{\beta}_{n-\nu-1}s^{n-\nu-1}+\cdots+\overline{\beta}_0)\cdot p_N(s;\tau)\end{aligned}$$

which are called 'vertex polynomials' of the polytope Ω, and the polytope Ω is the convex hull of them. Let the set of those vertex polynomials as

$$\Delta(s;\tau):=\{\delta_i(s;\tau):i=1,2,\cdots,m\}.$$

With $\delta_i(s;\tau)$, $\delta_j(s;\tau)\in\Delta(s;\tau)$, $1\le i,j\le m$, $i\neq j$, and $\lambda\in[0,1]$, let us call $\lambda\delta_i(s;\tau)+(1-\lambda)\delta_j(s;\tau)$ by an 'edge polynomial' of the polytope Ω.

Example 1: Consider an uncertain polynomial $p(s)=\alpha_2 s^2+\alpha_1 s+\alpha_0$, where uncertain coefficients $\alpha_2\in[1,2]$, $\alpha_1\in[3,4]$, and $\alpha_0\in[-1,3]$. In this case, the polytope Ω can be expressed as $\{p(s)=\alpha_2 s^2+\alpha_1 s+\alpha_0:\alpha_2\in[1,2],\alpha_1\in[3,4],\alpha_0\in[-1,3]\}$. There are $2^3=8$ vertex polynomials for this example. If we represent each polynomial in Ω as a point in the coefficient space, which is three-dimensional space, the vertex polynomials can be represented as a black dot in Fig. 3. There are $\binom{8}{2}=28$ edge polynomials, which are line

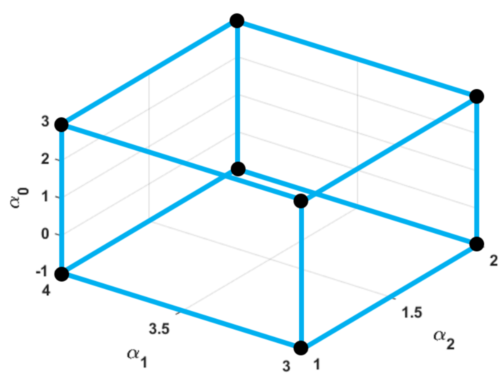

Fig. 3. The vertex polynomials (black dots) and exposed edge polynomials (sky-blue line segments) of the polytope Ω in the coefficient space.

segments connecting each pair of black dots in Fig. 3. Out of 28 edge polynomials, only 12 edge polynomials are drawn in sky-blue color in Fig. 3, which are called 'exposed edge polynomials'. For a formal definition of the exposed edge polynomial, refer to [14]. □

Lemma 3: The closed-loop system in Fig. 1 is robustly internally stable if and only if for each pair $\delta_i(s;\tau)$, $\delta_j(s;\tau) \in \Delta(s;\tau)$, $1 \le i, j \le m$, $i \neq j$, and for each $\lambda \in [0,1]$, the edge polynomial $\lambda\delta_i(s;\tau) + (1-\lambda)\delta_j(s;\tau)$ is Hurwitz. □

Proof: The lemma follows from the Edge theorem [14], which is stated as:

> Let $D \subset \mathbb{C}$ be a simply connected domain in the complex plane $\mathbb{C}$, and let Ω be a polytope of polynomials. Then the set of the roots of Ω
>
> $$R(\Omega) := \{s : f(s) = 0, f \in \Omega\} \subset \mathbb{C}$$
>
> is contained in D if and only if the collection of the roots of all the exposed edge polynomials of Ω is contained in D.

To prove Lemma 3, take D as the open left-half complex plane. If all the edge polynomials are Hurwitz, then all exposed edge polynomials are Hurwitz as well, and the sufficiency follows. The necessity is straightforward. ■

It is worthy to note that $\delta_i(s;\tau) \in \Delta(s;\tau)$ in Lemma 3 is no longer an uncertain polynomial. However, we still need to check infinitely many polynomials in terms of $\lambda \in [0,1]$ to decide the robust stability of the closed-loop system. The following lemma eliminates the λ-dependency in Lemma 3.

Before stating the next lemma, let us define the Hurwitz matrix of a polynomial. For a given polynomial $p(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ with real coefficients, the $n \times n$ matrix

$$H(p) = \begin{bmatrix} a_{n-1} & a_{n-3} & a_{n-5} & \cdots & \cdots & \cdots & 0 & 0 & 0 \\ a_n & a_{n-2} & a_{n-4} & & & & \vdots & \vdots & \vdots \\ 0 & a_{n-1} & a_{n-3} & & & & \vdots & \vdots & \vdots \\ \vdots & a_n & a_{n-2} & \ddots & & & 0 & \vdots & \vdots \\ \vdots & 0 & a_{n-1} & & \ddots & & a_0 & \vdots & \vdots \\ \vdots & \vdots & a_n & & & \ddots & a_1 & \vdots & \vdots \\ \vdots & \vdots & 0 & & & & a_2 & a_0 & \vdots \\ \vdots & \vdots & \vdots & & & & a_3 & a_1 & \vdots \\ 0 & 0 & 0 & & & & a_4 & a_2 & a_0 \end{bmatrix}$$

is called *Hurwitz matrix* of the polynomial $p(s)$. Moreover, when $a_n > 0$, the polynomial $p(s)$ is Hurwitz if and only if all the leading principal minors of the matrix $H(p)$ are positive [15]. Therefore, if $p(s)$ is Hurwitz, then $|H(p)| > 0$ so that $H(p)$ is invertible.

Lemma 4: The closed-loop system in Fig. 1 is robustly internally stable if and only if, for all $\delta_i(s;\tau) \in \Delta(s;\tau)$, the polynomial $\delta_i(s;\tau)$ is Hurwitz, and for each pair $\delta_i(s;\tau), \delta_j(s;\tau) \in \Delta(s;\tau)$, no eigenvalues of the matrix $H^{-1}(\delta_i(s;\tau))H(\delta_j(s;\tau))$ are located in the negative real axis $(-\infty, 0)$ in the complex plane. □

Proof: The proof uses Bialas' theorem [16], which is stated as follows:

> Let two polynomials with real coefficients
>
> $$f_1(s) = a_n^{(1)}s^n + a_{n-1}^{(1)}s^{n-1} + \cdots + a_0^{(1)},$$
> $$f_2(s) = a_n^{(2)}s^n + a_{n-1}^{(2)}s^{n-1} + \cdots + a_0^{(2)},$$
>
> where $a_n^{(1)}, a_n^{(2)} \neq 0$, are Hurwitz. Then, the convex combination $\lambda f_1(s) + (1-\lambda)f_2(s)$, where $\lambda \in [0,1]$, is Hurwitz if and only if no eigenvalues of $H^{-1}(f_1)H(f_2)$ are located in the negative real axis $(-\infty, 0)$, where H is the Hurwitz matrix.

In our case, the degree of each $\delta_i(s;\tau) \in \Delta(s;\tau)$ is determined only by the term $D(s)D_C(s)N_n(s)D_Q(s;\tau)$ in (3), and thus, its leading coefficient is always nonzero. Therefore, the degrees of all polynomials in the set $\Delta(s;\tau)$ are equal, and Bialas' theorem is ready to be applied. ■

Lemma 4 gives a necessary and sufficient condition on the robust stability of the DOB-based control system for a given τ, without the need to check infinitely many polynomials. Now, with the help of Routh-Hurwitz stability criterion, one can compute the exact value of τ^* and the detailed procedure is proposed in the next subsection.

B. Procedure for Computing τ^*

The following procedure is for both computation of τ^* when the plant set $\mathscr{P}$ in (1) consists of only minimum phase systems and choice of τ when the plant set $\mathscr{P}$ contains non-minimum phase systems.

Step 1. For each $\delta_i(s;\tau) \in \Delta(s;\tau)$, $i = 1, \cdots, m$, find the largest range $R_i \subset (0, \infty)$ such that for all $\tau \in R_i$, the polynomial $\delta_i(s;\tau)$ is Hurwitz.

In particular, if the plant set $\mathscr{P}$ consists of only minimum phase systems, existence of the largest τ_1^i (including the case when $\tau_1^i = \infty$) such that $(0, \tau_1^i) \subset R_i$ is guaranteed by Theorem 1. For the computation of R_i and τ_1^i, one may employ Routh-Hurwitz stability criterion (reviewed in the Appendix for convenience).

Step 2. For each pair $\delta_i(s;\tau), \delta_j(s;\tau) \in \Delta(s;\tau)$, obtain the largest range $R_{ij} \subset (0,\infty)$ such that for all $\tau \in R_{ij}$, no eigenvalues of $H^{-1}(\delta_i(s;\tau))H(\delta_j(s;\tau))$ are in $(-\infty, 0)$.

Similar to the previous step, existence of the largest τ_2^{ij} such that $(0, \tau_2^{ij}) \subset R_{ij}$ is also guaranteed by Theorem 1 when the plant set $\mathscr{P}$ consists of only minimum phase systems. For the computation of R_{ij} and τ_2^{ij}, one may employ Sturm's theorem, which is described in the Appendix.

Step 3. If the plant set $\mathscr{P}$ consists of only minimum phase systems, let $\tau^* = \min_{i,j}\{\tau_1^i, \tau_2^{ij}\} \le \infty$. Otherwise, let τ^* be the largest $\bar{\tau} \le \infty$ such that $(0, \bar{\tau}) \cap R^* = \emptyset$, where $R^* := (\bigcap_i R_i) \cap (\bigcap_{i,j} R_{ij})$ and $\emptyset$ denotes the empty set.

For the case where the plant set $\mathscr{P}$ contains non-minimum phase systems, Theorem 1 guarantees the existence of such $\bar{\tau}$. In fact, one can choose any $\tau \in R^*$ in order that the Q-filter-based DOB works for a given non-minimum phase plant. Even though R^* might be the empty set so that $\tau^* = \infty$ and the closed-loop system is not robustly internally stable for all $\tau > 0$, one can at least demonstrate if a given real plant set that contains non-minimum phase plant is suitable to employ the Q-filter-based DOB or not.

In the following section, illustrative examples that show the advantages of the proposed computation procedure are given.

V. Numerical Examples

In this section, two numerical examples are presented to describe the utility of the proposed computation procedure.

Example 2: It is assumed that in Fig. 1,

- $\mathrm{C}(s) = 2/(s+4)$,
- $\mathrm{P_n}(s) = 5/(s-2)$,
- $\mathrm{P}(s) = \beta_0/(s+\alpha_0)$, where $4 \le \beta_0 \le 10$, $-10 \le \alpha_0 \le 10$,
- $Q(s;\tau) = 1/(\tau s + 1)$.

It is obvious that the given $\mathscr{P}$ consists of only minimum phase systems since there is no zero-dynamics. Let us partially go through the computation procedure. There are four vertex polynomials,

$$\delta_1(s;\tau) = 5\tau s^3 + (-30\tau+4)s^2 + (-160\tau+8)s + 8,$$
$$\delta_2(s;\tau) = 5\tau s^3 + (70\tau+4)s^2 + (240\tau+8)s + 8,$$
$$\delta_3(s;\tau) = 5\tau s^3 + (-30\tau+10)s^2 + (-100\tau+20)s + 20,$$
$$\delta_4(s;\tau) = 5\tau s^3 + (70\tau+10)s^2 + (300\tau+20)s + 20.$$

First, the largest $\tau_1^1 > 0$ with which $\delta_1(s;\tau)$ is Hurwitz for all $\tau \in (0, \tau_1^1)$ is computed as 0.0457 by Routh-Hurwitz stability criterion. Secondly, the largest $\tau_2^{34} > 0$, such that for all $\tau \in (0, \tau_2^{34})$, no eigenvalues of $H^{-1}(\delta_3(s;\tau))H(\delta_4(s;\tau))$ are in $(-\infty, 0)$, is calculated as 0.1667 by Sturm's theorem. In fact,

$$\tau^* = \min_{i,j}\{\tau_1^i, \tau_2^{ij}\} = 0.0457.$$

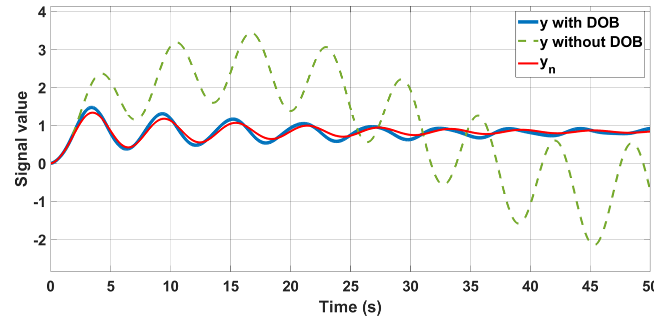

Fig. 4. Nominal performance recovery with $\mathrm{P}(s) = \frac{s^2-0.2s+5}{s^3+3s^2+3s+1}$ for $\tau = 0.21$.

The result can be verified by `wcgain` function (that calculates the worst-case peak gain of given uncertain system) in MATLAB and it is observed that for $\tau = 0.0458$, transfer functions of r to y and d to y can have infinite gain because of the plant uncertainty. On the other hand, gains of the same transfer functions for $\tau = 0.0456$, are bounded despite the plant uncertainty. □

Example 3: Suppose that in Fig. 1,

- $\mathrm{C}(s) = 1/(s+1)$,
- $\mathrm{P_n}(s) = (s^2+s+5)/(s^3+3s^2+3s+1)$,
- $\mathrm{P}(s) = (s^2+\beta_1 s+5)/(s^3+3s^2+3s+1)$, where $-0.2 \le \beta_1 \le 2$,
- $Q(s;\tau) = 1/(\tau s+1)$.

Although the given set $\mathscr{P}$ contains non-minimum phase systems, the procedure provides that the closed-loop system with Q-filter-based DOB is robustly internally stable at least for $\tau \in (0.206, 0.627) \subset R^*$.

For $r(t) = 1(t)$ (i.e., Heaviside step function) and $d(t) = 2\sin(0.1t)$, the following Fig. 4 and 5 show the stability of the closed-loop system and nominal performance recovery with two different real plant models for $\tau = 0.21$. It is observed that the closed-loop system is robustly stable regardless of non-minimum phaseness of the system and the nominal performance recovery is achieved to some extent. On the other hand, if the value of τ gets smaller and even out of the given range $(0.206, 0.627)$, we expect that there exist some $\mathrm{P}(s) \in \mathscr{P}$, such that the closed-loop system with $\mathrm{P}(s)$ is not internally stable. As was expected, Fig. 6 demonstrates that the closed-loop system with $\mathrm{P}(s) = (s^2-0.2s+3)/(s^3+3s^2+3s+1)$ is unstable for $\tau = 0.16$. □

VI. Concluding Remarks

In this paper, several lemmas on necessary and sufficient condition for robust stability of the DOB-based control system were presented and the design of the Q-filter-based DOB including computation of minimum bandwidth of the Q-filter was proposed. Additionally, it was noted that, contrary to the past belief, we can exploit the existing Q-filter-based DOB scheme as it is with non-minimum phase plants even without modifying the standard structure of the Q-filter-based DOB.

In fact, if the computation procedure introduced in Section IV is implemented impeccably, then a tool that gives the exact value of τ^* for a given system environment can be

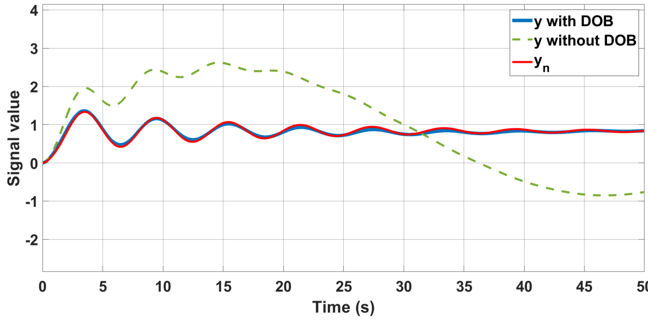

Fig. 5. Nominal performance recovery with $\mathrm{P}(s) = \frac{s^2+2s+5}{s^3+3s^2+3s+1}$ for $\tau = 0.21$.

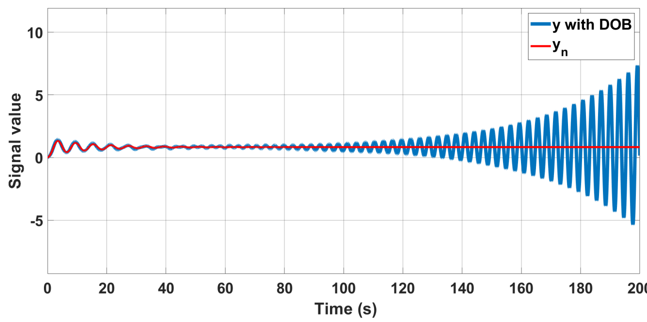

Fig. 6. Unstable closed-loop system with $\mathrm{P}(s) = \frac{s^2-0.2s+5}{s^3+3s^2+3s+1}$ for $\tau = 0.16$.

suggested as in [17], where a MATLAB toolbox named 'DO-DAT' is introduced. Some of the main results of this paper will be included in the feature of DO-DAT as soon as possible.

APPENDIX

Routh-Hurwitz stability criterion [18]: Let $p(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ be a polynomial of degree n. The Routh-Hurwitz table of $p(s)$ can be made up as follows:

a_n	a_{n-2}	a_{n-4}	$\cdots$
a_{n-1}	a_{n-3}	a_{n-5}	$\cdots$
b_1	b_2	b_3	$\cdots$
c_1	c_2	c_3	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$

where

$$b_i = \frac{a_{n-1}a_{n-2i} - a_n a_{n-(2i+1)}}{a_{n-1}}, \; c_i = \frac{b_1 a_{n-(2i+1)} - a_{n-1}b_{i+1}}{b_1},$$

for $i = 1, 2, \cdots$. Then, the number of sign changes in the first column of the Routh-Hurwitz table of $p(s)$ is equal to the number of roots with non-negative real part of $p(s)$.

Sturm's theorem [19]: Let $p(s)$ be a polynomial with real coefficients and define

$$p_0(s) := p(s),$$
$$p_1(s) := p'(s),$$
$$p_{i+1}(s) := -\mathrm{rem}(p_{i-1}(s), p_i(s)), \qquad i = 1, 2, \ldots$$

where $p'(s)$ is the derivative of $p(s)$ and $\mathrm{rem}(p_{i-1}(s), p_i(s))$ represents the remainder of the division of $p_{i-1}(s)$ by $p_i(s)$. Then, the sequence of polynomials $p_0, p_1, \ldots$ is called *Sturm sequence* of $p(s)$, which is a finite sequence. Let $\#(\zeta, p)$ be the number of sign changes in the Sturm sequence of $p(s)$ at $s = \zeta \in \mathbb{R}$. Then, the number of distinct real roots of $p(s)$ in the interval $(a, b]$ of the real axis is equal to

$$\#(a, p) - \#(b, p).$$

In order to apply Sturm's theorem for the interval $(-\infty, 0)$, one has to compute $\#(-\infty, p)$. The sign of a polynomial $p(s)$ at $s = -\infty$ is defined as the sign of the leading coefficient, if $p(s)$ has even degree, and the opposite sign of the leading coefficient, if $p(s)$ has odd degree.

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